Erdős's conjecture on the pancyclicity of Hamiltonian graphs

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An n-vertex graph is *Hamiltonian* if it contains a cycle covering all its vertices and it is *pancyclic* if it contains cycles of all lengths from 3 up to n. In 1973, Bondy stated his celebrated meta-conjecture that any non-trivial condition which implies that a graph is Hamiltonian should also imply that it is pancyclic (up to a certain collection of simple exceptional graphs). As an example, consider the classical Dirac's theorem stating that every n-vertex graph with minimum degree at least $n/2$ is Hamiltonian. Strengthening this, Bondy himself showed that every such graph is in fact either pancyclic or isomorphic to the complete bipartite graph $K_{n/2,n/2}$.

Bondy's meta-conjecture deals with conditions for Hamiltonicity which imply pancyclicity. In a similar fashion, one can ask the following natural question: Let G be a Hamiltonian graph; under which assumptions can we guarantee that G is also pancyclic? Indeed, also in the 1970s, Erdős put forward the problem below.

Problem. Given an n-vertex Hamiltonian graph with independence number $\alpha(G) \leq k$, how large does n have to be in terms of k in order to guarantee that G is pancyclic?

He proved that it is enough to have $n = \Omega(k^4)$ and conjectured that already $n = \Omega(k^2)$ should be enough - a simple construction shows that this is best possible. Since then there have been several improvements of Erdős's initial result – by Keevash and Sudakov who proved that $n = \Omega(k^3)$ is enough, by Lee and Sudakov who improved it to $n = \Omega(k^{7/3})$, and finally by Dankovics who showed that $n = \Omega(k^{11/5})$ suffices. We resolve the conjecture of Erdős in a strong form, showing that if a Hamiltonian graph G has $n \geq 2k^2 + o(k^2)$ vertices and $\alpha(G) \leq k$, then G is pancyclic. This is asymptotically best possible.

This is joint work with Nemanja Draganić and Benny Sudakov.