There are at most $O(m^{0.31m})$ non-isomorphic combinatorial 3-spheres with m faces

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How many combinatorial 3-spheres are there with m facets? That is, how many simplicial complexes with m maximal faces are there whose geometric realizations are homeomorphic to the unit sphere in Euclidean 4-space?

This was one of Gromov's long list of questions [Gromov, 1999], and he conjectured that there are at most $exp(O(m))$ combinatorially distinct d-spheres for any d (i.e., distinct up to a permutation of the vertex set). The previous best known upper bound was m^{cm} with $c = 1/3 + o(1)$ [Rivasseau, 2013], obtained by studying certain topological properties of the dual graph. We improve this to $c < 0.31$ by instead using an information-theoretic approach.

We study the set C_m of 3-spheres (with $\leq m$ faces) that are minimal with respect to a certain collection of Pachner-type moves. First, we use an entropy counting lemma from a recent paper [Palmer & Pálvölgyi, 2020] — which is essentially a clever reformulation of Shannon's noiseless encoding theorem [Shannon, 1948] — to upper bound the size of C_m. We then show that for an arbitrary 3-sphere with m faces, there is a sequence of Pachner moves leading to it from some sphere in C_m, and which can be specified with $O(m)$ bits.