The threshold for stacked triangulations

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Consider a bootstrap percolation process that starts with an initial set of 'infected' triangles $Y \sim Y(n,p)$, where each of the $\binom{n}{3}$ triangles with vertices $[n] = \{1, 2, \ldots, n\}$ appears independently with probability p. Then, a new triangle f gets infected if there is a copy of K_4^3 (= the boundary of a tetrahedron) in which f is the only not-yet infected triangle.

What is the critical probability for percolation — the event that all triangles get infected? How many new triangles do get infected in below this threshold?

Equivalently, a stacked triangulation is obtained by repeatedly subdividing a triangle into 3 new triangles. The above questions would amount to asking, e.g., about the critical probability so that the random simplicial complex $Y(n,p)$ would typically contain, for each of the $\binom{n}{3}$ triangles $\{a, b, c\}$, the faces of a stacked triangulation whose internal vertices are labelled in $[n]$ and its boundary is labelled a, b, c.

We consider these questions for triangulations in every dimension $d \geq 2$, and our main result identifies a sharp probability threshold for percolation, showing it is asymptotically $(c_d * n)^{(-1/d)}$, where c_d is the growth rate of the Fuss–Catalan numbers of order d.

The proof hinges on a second moment argument in the supercritical regime, and on Kalai's algebraic shifting in the subcritical regime.

Joint work with Eyal Lubetzky.