The Potential Function Approach for Maker-Breaker Games

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Abstract

Maker-Breaker games are sequential games with perfect information played by two players, called Maker and Breaker. For $n, q \in \mathbb{N}$ and a subgraph C of the complete graph K_n the Maker-Breaker C-game is played as follows. Alternately Maker claims an edge and thereafter Breaker claims up to q edges of K_n until every edge has been claimed by exactly one of the two players. If at the end of the game the graph that consists of Maker's edges contains a copy of C, Maker wins the game, otherwise Breaker wins. Bednarska and Łuczak (1999) showed that there exist constants c_0 and c_1 such that for large enough n Maker has a winning strategy if $q \leqslant c_0 n^{1/m(C)}$ and Breaker has a winning strategy if $q \geqslant c_1 n^{1/m(C)}$ where $m(C) = \max_{H \subseteq C, |V(H)| \geqslant 3} \frac{|E(H)|-1}{|V(H)|-2}$. They also conjectured that these constants could be chosen arbitrarily close to each other. Not only is this conjecture open for every graph C which contains a cycle, but also for the most graphs there are no good constants known. In the most studied case, where C is a cycle of length 3, Chvátal and Erdős (1978) showed that for large n Maker can win the game if $q \leqslant \sqrt{2}\sqrt{n}$ and Breaker can win if $q \geqslant 2\sqrt{n}$. Today the lower bound of $\sqrt{2}$ is still the best known and it took a long time for the first slight improvement of the upper bound from 2 to 1.958 by Balogh and Samotij (2011) with a randomized Breaker strategy. In a recent breakthrough work Glazik and Srivastav (2022) used the idea of a potential function to reduce the upper bound significantly to $\sqrt{8/3} \approx 1.633$. In this work we present a general sufficient criterion for a Breaker win with the help of potential functions and apply it to Maker-Breaker C-games where C is a cycle of arbitrary length, a class of games for which no good constants were known previously.
(joined work with Anand Srivastav)