Cycles in Mallows random permutations

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We study cycle counts in permutations of $1, \ldots, n$ drawn at random according to the Mallows distribution. Under this distribution, each permutation $\pi \in S_n$ is selected with probability proportional to $q^{\mathrm{inv}(\pi)}$, where $q > 0$ is a parameter and $\mathrm{inv}(\pi)$ denotes the number of $i < j$ such that $\pi(i) > \pi(j)$. For ℓ fixed, we study the vector $(C_1(\Pi_n), \ldots, C_\ell(\Pi_n))$ where $C_i(\pi)$ denotes the number of cycles of length i in π and Π_n is sampled according to the Mallows distribution.

Here we show that if $0 < q < 1$ is fixed and $n \to \infty$ then there are positive constants m_i such that each $C_i(\Pi_n)$ has mean $(1 + o(1)) \cdot m_i \cdot n$ and the vector of cycle counts can be suitably rescaled to tend to a joint Gaussian distribution. Our results also show that when $q > 1$ there is a striking difference between the behaviour of the even and the odd cycles. The even cycle counts still have linear means, and when properly rescaled tend to a multivariate Gaussian distribution. For the odd cycle counts on the other hand, the limiting behaviour depends on the parity of n when $q > 1$. Both $(C_1(\Pi_{2n}), C_3(\Pi_{2n}), \ldots)$ and $(C_1(\Pi_{2n+1}), C_3(\Pi_{2n+1}), \ldots)$ have discrete limiting distributions – they do not need to be renormalized – but the two limiting distributions are distinct for all $q > 1$. We describe these limiting distributions in terms of Gnedin and Olshanski's bi-infinite extension of the Mallows model.

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